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B.Sc. Sem-II MJC-02

(1)

Que (1) :– L'Hospital's Rule for the Fundamental form

Let $\phi(x)$ and $\psi(x)$ be functions of x such that and $\phi'(a)$, $\psi'(a)$ both exist and $\psi'(a) \neq 0$

then $\lim_{x\to a} \dfrac{\phi(x)}{\psi(x)} = \dfrac{\phi'(a)}{\psi'(a)}$

Proof:– Since $\phi(x)$ and $\psi(x)$ are differentiable finitely at a, they are Continuous at a and so

$$\phi(a) = \lim_{x\to a} \phi(x) = \lim_{h\to 0} \phi(a+h) = 0 \quad \text{— (1)}$$

and $$\psi(a) = \lim_{x\to a} \psi(x) = \lim_{h\to 0} \psi(a+h) = 0 \quad \text{— (2)}$$

Now, $$\phi'(a) = \lim_{h\to 0} \frac{\phi(a+h) - \phi(a)}{h}$$

$$= \lim_{h\to 0} \frac{\phi(a+h)}{h} \quad \text{From (1)}$$

and $$\psi'(a) = \lim_{h\to 0} \frac{\psi(a+h) - \psi(a)}{h}$$

$$= \lim_{h\to 0} \frac{\psi(a+h)}{h}, \text{ by (2)}.$$

Hence by the theorem on limit (Provided $\psi'(a) \neq 0$)

We get $$\frac{\phi'(a)}{\psi'(a)} = \lim_{h\to 0} \frac{\phi(a+h)/h}{\psi(a+h)/h} = \lim_{h\to 0} \frac{\phi(a+h)}{\psi(a+h)}$$

$$= \lim_{x\to 0} \frac{\phi(x)}{\psi(x)}$$

General Form :– If $\phi(a) = \phi''(a) = \dots\dots = \phi^{n-1}(a) = 0$

$\psi'(a) = \psi''(a) = \dots\dots = \psi^{n-1}(a) = 0$

but $\phi^n(x)$ and $\psi^n(x)$ are not both zero as $x\to a$

then in that Case

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi^n(x)}{\psi^n(x)}$$

Note :– The proposition of this article holds even. If we have ∞ instead of a.

Let $x = \frac{1}{t}$

If $\lim_{x\to\infty} \phi(x) = 0$ and $\lim_{x\to\infty} \psi(x) = 0$

then $\lim_{x\to\infty} \frac{\phi(x)}{\psi(x)} = \lim_{t\to 0} \frac{\phi\left(\frac{1}{t}\right)}{\psi\left(\frac{1}{t}\right)} = \lim_{t\to 0} \frac{\phi'\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)}{\psi'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}$

$$= \lim_{t\to 0} \frac{\phi'\left(\frac{1}{t}\right)}{\psi'\left(\frac{1}{t}\right)} = \lim_{x\to\infty} \frac{\phi'(x)}{\psi'(x)}$$

Que ② :— $\lim_{x\to 0} \frac{\log(1+kx^2)}{1-\cos x}$

Soln :— limit $= \lim_{x\to 0} \frac{\log(1+kx^2)}{1-\cos x}$ [Form $\frac{0}{0}$]

$$= \lim_{x\to 0} \frac{\frac{1}{1+kx^2}\cdot 2kx}{\sin x} = \lim_{x\to 0} \frac{2kx}{(1+kx^2)\cdot\sin x} \quad \left[\text{Form } \tfrac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2k}{2kx\cdot\sin x + (1+kx^2)\cdot\cos x} = 2k.$$

Que ③ :— Find the limiting value of $\frac{1}{x} - \cot x$, when x tends to zero.

Soln :— It is in the form $\infty - \infty$

$\therefore$ limit $= \lim_{x\to 0}\left(\frac{1}{x} - \frac{\cos x}{\sin x}\right)$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x\sin x}, \quad \left[\text{Form } \tfrac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x - x\cos x)}{\frac{d}{dx}(x\sin x)}$$

$$= \lim_{x\to 0} \frac{\cos x - \cos x + x\sin x}{\sin x + x\cos x} \quad \left[\text{Form } \tfrac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(x\sin x)}{\frac{d}{dx}(\sin x + x\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x + x\cos x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$$